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Application of the W-transformation to compute the linearised 2-d beach problem potentials

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Abstract

The computation of the 2-d beach problem is reconsidered. A method which allows both regular and singular components to be evaluated for arbitrary values of beach slopes through direct application of the W-transformation to a divergent form of the solution integral is established. Previous models were restricted to the very simple beach angles $\alpha = \pi/2k,\ k \in \mathbf{N}$. Results indicate that the procedure is sufficiently fast that these potentials (so computed) may be used as kernels in future Green function computations.

MSC: 65N; 74J

Keywords: Infinitessimal waves; Potential flow; Wedge domain; W-transformation

1. Introduction

Solutions to the 2-d linearised plane beach wave problem have a history dating back to very late nineteenth century. The earliest exposition by Kirchoff discusses the finite bounded solution for a 45° bed. Much interest then stirred by e.g. Weinstein [22], Stoker [21] and Friedrichs [8] in the late 1940s, led to a plethora of publications in the ensuing decade notably [10,12–14,17]. Understandably, in view of the computational facilities at the time, none of these works seriously tackle the problem of computation in other but the simplest of cases where $\alpha = \pi/2k,\ k \in \mathbf{N}$. Thus, the works [21,8] for example displayed graphs of the potential for the cases $k = 2,\ k = 15$, although in both cases certain graphs are in error (see [2]). In the work [2], the author rectified these errors and in a subsequent series of papers e.g. [4] or [7] many further computations were made but always for the special slope angles.

More recently, in [6], the author developed a routine for computing the bounded wave solution, for general beach slopes, in the case of obliquely incident waves. However, the problem of computing the second linearly independent solution (of the first kind), which is logarithmically singular at the shore-line, was not tackled for these general angles. For the very simple beach angles, this was however, computed in [5].

Missing therefore, is an exposure detailing computation of the singular solution in the cases where beach angles are other than very simple. For steep beaches, or for certain shore protection schemes, this could be particularly important, as there are only one or two possible values of $k \in \mathbf{N}$ that give significantly steep beaches. In the present note this is discussed for the case of normal incidence (i.e. the 2-d case). Indeed, both the regular and singular waves will be

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computed (in contrast to previous methods there is no extra effort required in the method proposed here) and, moreover the limit of normal incidence is not easily taken in the 3-d version for the regular (bounded) wave given in [6].

One of the main features of the computation is the use of direct integration of a highly oscillatory infinite integral (an inverse Mellin transform) and this is facilitated by invoking the W-transformation pioneered by Sidi [18] and developed further in [19,20]. In fact, it turns out that in order to optimize the evaluation of the integrand (also through an infinite integral), it is necessary to carry out the computation on a contour on which the integral actually diverges. In previous similar computations for simple beaches e.g. [2], it had been the practice to subtract the far-field asymptotic form of the wave to yield an absolutely convergent integral. The use of the W-transformation disposes of a need for this and very accurate results are now obtained with the integral in divergent form. This is substantiated by comparison with previous data available only for cases of simple beach angles. Moreover, the computation is sufficiently fast that it may be envisaged as a Green's function computation for further use in the solution of a range of problems in wedge-shaped domains. Work is currently in progress on such problems.

The potentials are lifted from previous works and presented in the next section. Then follows a discussion on the computation details of the inner integral with the details on the strategy for the outer integral presented in Section 4. As an aid to other researchers a f77-code[1] is provided through a web-link http://www.city.londonmet.ac.uk/cismres/xtra/test.txt. This is the code used in the calculation of all results presented in the concluding Section 5.

2. The problem and its solution

Polar coordinates r, θ are used with the polar line $r=0$ representing the shore line, $\theta=0$ the *SWL* and $\theta=-\alpha$ the rigid bottom so that the wedge so formed represents the primary domain D of the flow. Periodic wave motion at angular frequency ω is assumed enabling the problem to be non-dimensionalised w.r.t. the length g/ω^2 and the time ω^{-1}. It is assumed that the non-dimensional velocity potential Φ is given by

$$\Phi = \Re(\phi(R,\theta)\mathrm{e}^{\mathrm{i}\omega t}),$$

where $R=(\omega^2/g)r$. The relevant equations are

$$\nabla^2\phi = 0, \quad (R,\theta)\in \mathrm{D}, \tag{1}$$

$$\frac{\partial\phi}{\partial\theta} = 0, \quad \theta=-\alpha, \tag{2}$$

$$\frac{1}{R}\frac{\partial\phi}{\partial\theta} = \phi, \quad \theta = 0, \tag{3}$$

$$\phi/\log R \quad \text{bounded as } R\to 0, \tag{4}$$

$$\phi\to\Phi_\infty, \quad R\to\infty, \tag{5}$$

where Φ_∞ is a suitable 'deep-water' potential. The solution to the problem is given by the basis $\{\phi_r, \phi_s\}$ (see, e.g. [4] Eqs. (2.8)–(2.11)), where

$$\phi_r = \phi_r^{(0)} - \sin\chi \mathrm{e}^{R\sin\theta}, \quad \phi_s = -\phi_s^{(0)} + \cos\chi \mathrm{e}^{R\sin\theta}. \tag{6}$$

Here $\chi = R\cos\theta+\gamma$, $4\gamma = \pi(1+k)$, and

$$\phi_r^{(0)} = \frac{1}{2\pi \mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \Gamma(s)R^{-s}\left\{\sin[s(\theta+\pi/2)+\gamma] - \frac{B_k(s)\sin\pi s\cos s(\theta+\alpha)}{\sqrt{(2\pi)}\cos s\alpha}\right\}\mathrm{d}s, \tag{7}$$

$$\phi_s^{(0)} = \frac{1}{2\pi \mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \Gamma(s)R^{-s}\left\{\cos[s(\theta+\pi/2)-\gamma] + \frac{B_k(s)\cos\pi s\cos s(\theta+\alpha)}{\sqrt{(2\pi)}\cos s\alpha}\right\}\mathrm{d}s, \tag{8}$$

[1] This code may be fragile on some compilers. It was run here on the public domain g77 available in the djgpp (slatec) wrapping.

where $B_k(s)$ is defined by

$$B_k(s) = \Gamma(s)\exp\left[\int_0^\infty \frac{\mathrm{d}t}{t}\left\{\frac{2\mathrm{e}^{t/2}\sinh(s-\frac{1}{2})t}{(\mathrm{e}^{kt}+1)(\mathrm{e}^t-1)} - \left(s-\frac{1}{2}\right)\mathrm{e}^{-t}\right\}\right], \quad -k < \Re s < k+1. \tag{9}$$

In the above it is clear that to compute $B(s)$ economically it will prove convenient to take $c=\frac{1}{2}$. Note that there is no assumption that $k \in \mathbf{N}$ and note also the double pole at $s=0$ in Eq. (8) which ensures the logarithmic singularity of ϕ_s at $R=0$.

2.1. Convergence

The asymptotics of the braces in Eqs. (7) and (8) above have been thoroughly investigated in previous works, e.g. [2,3] and with $s=c+\mathrm{i}\tau$, the combination $\Gamma(s)\times\{\cdot\}$ decays like $\exp(-\pi\tau/k)$, whilst, of course, $|R^{-s}|=1/\sqrt{R}$. Thus, it is evident that, relying on this exponential decay, shallower beaches appear to present more of a challenge numerically.

By far the greatest complication however, arises from the interfering oscillatory components in the composite expression. Both $\Gamma(s)$ and R^{-s} oscillate (the former with increasing frequency) and furthermore $B_k(s)$ also has a phase function which varies on the line of integration. Fortunately, as will be seen in the next section that oscillation is attenuated by a factor $\exp(-\pi\tau/k)$.

It is perfectly possible, however, to replace the absolutely convergent form of inversion equations (7) and (8) by the simpler pair

$$(-\phi_r, \phi_s) = \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \Gamma(s)R^{-s}\left\{\frac{B_k(s)(\sin\pi s, \cos\pi s)\cos s(\theta+\alpha)}{\sqrt{(2\pi)}\cos s\alpha}\right\}\mathrm{d}s, \tag{10}$$

where $\int\!\!\!\!\to$ denotes that the integral is examined in the sense of Abel summability (see e.g. [9, p. 136]); and this is the procedure explored in the present work. Some explanation of this is deferred to the Appendix.

3. The numerical ansatz for $B_k(s)$

Maliuzhinets [16,15] defined a function, now known as Maliuzhinets' function $M_\beta(s)$, as a solution $f=M_\beta$ of

$$\frac{f(s+2\beta)}{f(s-2\beta)} = \cot\left\{\frac{1}{2}\left(s+\frac{1}{2}\pi\right)\right\} \tag{11}$$

which is analytic in $-2\beta-\pi/2 < \Re(s) < 2\beta+\pi/2$ and satisfies $f(0)=1$. Because $B_k(s+1)=B_k(s)\tan s\alpha$ (e.g. [2]), it is clear that $B_k(s)$ as defined by Eq. (9) is related to Maliuzhinets' functions and after a small amount of manipulation using Kummer's result (see [23, p. 250]), we can express this relationship by

$$B_k(s) = \left\{\frac{\pi}{\sin\pi s}\,\frac{M_\beta(\frac{\pi}{2}+(s-\frac{1}{2})4\beta)}{M_\beta(\frac{\pi}{2}-(s-\frac{1}{2})4\beta)}\right\}^{1/2}, \quad \beta=\pi/4k, \quad 0<\Re(s)<1. \tag{12}$$

Expression (9) is not particularly well suited for computation on the line chosen, nor (to the best of the author's knowledge) are there any readily available routines for computation of Maliuzhinets' functions. Consequently, a strategy for computing B_k needs to be considered.

Note that B_k satisfies the same 'folding' formula as the gamma function, namely

$$B_k(s)B_k(1-s) = \frac{\pi}{\sin\pi s} \tag{13}$$

from which there readily follows

$$|\{\Gamma(s), B_k(s)\}|^2 = \frac{\pi}{\cosh\pi\tau}.$$

It remains to assess $\arg B_k(s)$. Correction of a sign in the primary expression of the corollary to Appendix 1 in [3] leads to the following result

$$\arg B_k(s) = -\frac{\pi k}{4} + \frac{e^{-\pi\tau/k}}{\sin\alpha} - \frac{e^{-2\pi\tau/k}}{2}\Im\int_0^\infty \frac{e^{2i\tau t}\tanh kt}{(t+i\pi/k)\sinh(t+i\pi/k)}\,dt.$$

One can compute this (inner) integral using the routine *dqawf* (essentially a Fourier transform integrator) from the QUADPACK suite. Note also that, whilst the oscillations in calculated values of B_k are bound to be attenuated as $\tau \to \infty$ if k is substantial, the function will nevertheless undergo a considerable number of oscillations. In particular the complex number B_k will spin from zero argument at $\tau = 0$ to a final value of $-\pi k/4$ at infinity. The formula given shows that this spinning is essentially ended once $(e^{-\pi\tau/k})/\sin\alpha < \pi$ and that can be guaranteed if $\tau > (k/\pi)\log|2k/\pi^2|$.

4. The outer integration

An alternative numerical integration procedure is to make use of Sidi's extrapolative routine [20] in which the W-algorithm is designed to exploit the regular oscillatory nature of the integrand. This has been previously used by the author in similar computations, see for example [6] for calculation of the 3-d bounded wave. This will now be used in the exterior computation strategy and it is noted that we can then treat the whole solution directly since Sidi's method works also for integrals that only converge weakly as Cauchy Principal Value or in the summability sense of Abel. This would be the case with the full form given as Eq. (10). Because of conjugacy, these are equivalent to the simpler (semi-infinite) forms

$$\phi_r = -\frac{1}{\pi}\Re \int_0^{\infty} \Gamma(s)R^{-s}\left\{\frac{B_k(s)\sin\pi s\cos s(\theta+\alpha)}{\sqrt{(2\pi)}\cos s\alpha}\right\}d\tau, \tag{14}$$

and

$$\phi_s = \frac{1}{\pi}\Re \int_0^{\infty} \Gamma(s)R^{-s}\left\{\frac{B_k(s)\cos\pi s\cos s(\theta+\alpha)}{\sqrt{(2\pi)}\cos s\alpha}\right\}d\tau, \tag{15}$$

where $s = \frac{1}{2} + i\tau$. The option to exploit the oscillatory nature of these integrals is justified on two counts. Iserles [11] has recently shown that, contrary to popular conception, high oscillation is actually beneficial to economic computation and, moreover, as stated earlier, the asymptotics of the tail of the integrand behaves like $\exp(-\pi\tau/k)$ and therefore, weakens the rate of convergence for shallower beaches. The equivalent effect (by linear scaling) on an oscillatory integrand is to increase the oscillation thus making the subsequent numerical strategy more robust for these shallower beaches. These remarks are based on the earlier observations of Sidi (see [18,19]) which establish that there is no significant reduction in the performance of oscillatory integrals which decay slowly or even those that do not decay at all but are instead summable in the sense of Abel.

The procedure now is first to integrate from 0 to τ_0 where τ_0 is chosen so that $B_k(s)$ has stopped spinning whilst simultaneously the oscillations in $\Gamma(s)$ have begun to dominate those of R^{-s}. This can be done with the routine *dqag* from QUADPACK and it will ensure that, in the region $\tau > \tau_0$ where the W-transformation is to be applied, all oscillation is controlled by $\Gamma(s)R^{-s}$.

The usefulness of the W-transformation was highlighted in [20] where the author showed that explicit asymptotic analysis of the integrand tail was not necessary and could be replaced without loss of accuracy by a few more iterations. Coupled with the further observation that it was not necessary to calculate zeros of the phase function and that instead any fixed phase could be used, the method had now become very competitive. The last point meant that complex forms could be used without splitting into sines and cosines thus providing a further saving in labour.

4.1. *The numerical details and the f77 code*

With the help of Stirling's formula for the gamma function is identified a suitable τ_0 by the selection $\tau_0 = \max(k/\pi\log|2k/\pi^2|, 10eR)$. The governing part of the phase function in the integrand is seen to be $\tau\log(\tau/eR)$. An integer N_0 is then defined by $N_0 = \tau_0\log(\tau_0/eR)$ so that the further values of τ with the fixed phase thus defined are given by solutions to $\tau = R\exp(1 + (N_0 + j)\pi/\tau)$, $j = 1, 2, \ldots, Npoints$ where *Npoints* represents the number of iterations in the W-algorithm [20]. These points τ_j are determined by Newton–Raphson iterations.

The f77 coding can be found at http://www.city.londonmet.ac.uk/cismres/xtra/test.txt and requires linking with elements from the public domain QUADPACK (available, for example, in the 'slatec' wrapping). In particular *dqawf*, *dqag* are used in, respectively, the inner and outer integrations but these routines also call a number of related subroutines. The gamma function is calculated using Stirling's formula with a continued fraction representation of the tail. The W-algorithm is coded as described in the series of papers by Sidi.

5. Results and conclusions

Results are presented for two 'design' beaches. The values $\alpha = \pi/4$, $\pi/30$ allow both a steep and relatively shallow beach to be examined. The choice also means that, at least for the regular wave, a comparison with an exact closed-form solution is possible. In each case, the regular (a) and singular (b) wave components calculated as described herein are subtracted from their counterparts computed for (a) from the expression given originally by Stoker [21] only for special slope angles of this type and for (b) using the method described in e.g. [4] where absolute convergence is used but without the W-algorithm. The resultant error curves are then displayed on a R-grid from $R = 1$ to $R = 20$. Note that the vertical scale in Fig. 1 is different to that in Figs. 2 and 3.

It is clear from the results that:

1. The present method compares very favourably with exact evaluation of the regular wave (error nowhere greater than 10^{-9}).
2. The W-algorithm applies very well to Abel-summable integrals and, in particular, its implementation is a practical option in a real application.
3. The method here identifies no difference in computing the regular and singular waves. Thus, the results for the singular wave are as accurate as those for the regular wave. Because the errors on the singular graphs are greater, it means that previous singular wave computations were less accurate than their regular wave counterparts.
4. The computations are done in a reasonable time; here 382 potential function values are computed with the above error constraint in about 3 min using a 2004 model Toshiba laptop with a Pentium(R) 1.7 GHz processor.

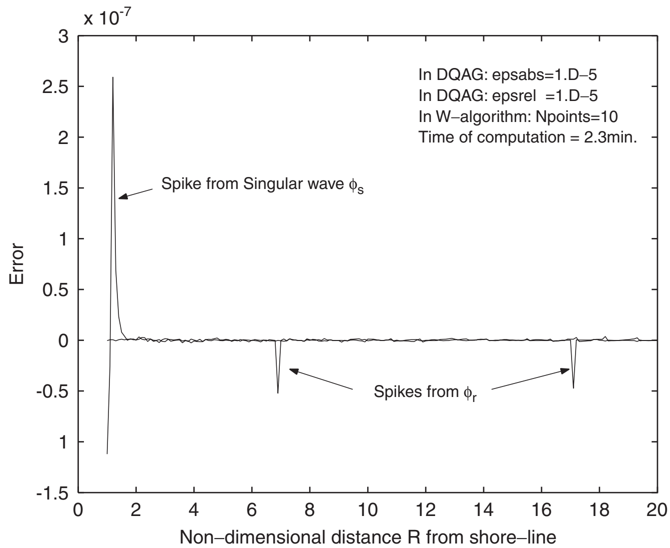

Fig. 1. Error computation for present method applied to regular and singular waves. Note (by comparison with Figs 2,3) how low error demand creates spikes.

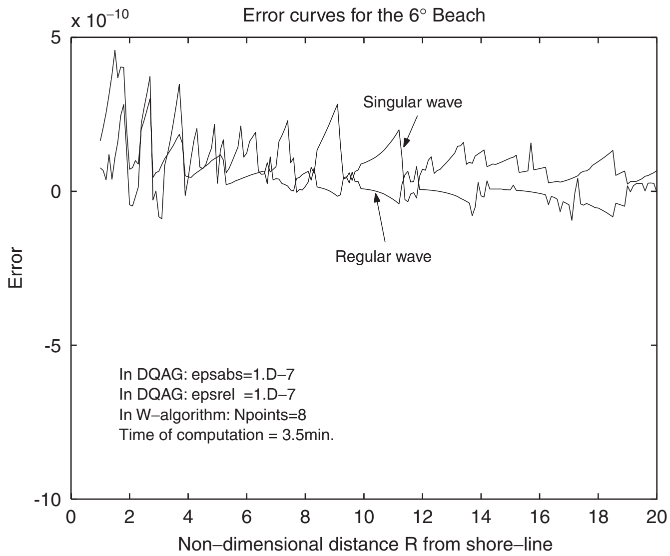

Fig. 2. Error computation for present method applied to regular and singular waves on a 6° beach. Note higher error demand.

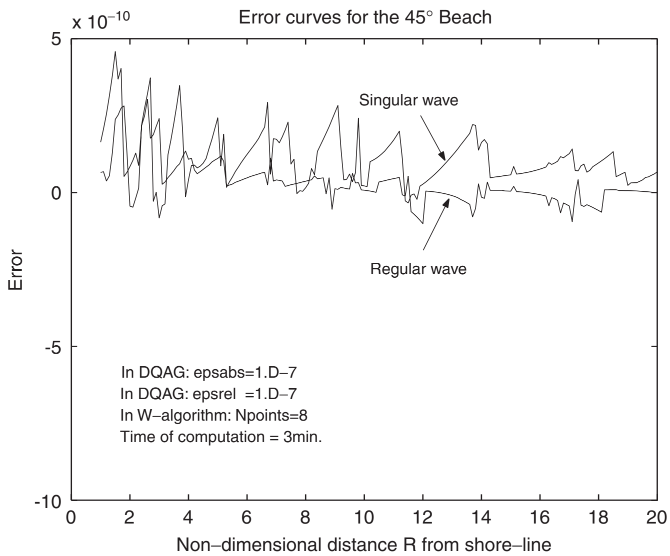

Fig. 3. Error computation for present method applied to regular and singular waves on a 45° beach.

5. From Fig. 1 it is clear that a significant time saving is achieved by relaxing the error demands placed in the DQAG routine and on the subsequent results. This would be important for a full water column representation or where Greens function calculation is required.

The challenge now is to develop a feasible speedy method of calculating the singular wave component in the oblique incidence problem. As stated earlier, presently such computation only exists for the regular wave [6] or the singular wave on very simple beaches [5].

Appendix A. Development of solution as an Abel summable integral

To simplify this discussion take $\theta = 0$ because this is the only value of θ for which the notion of Abel summability is necessary. Consider the regular wave (the argument follows identically for the singular wave). From Eq. (7),

$$\phi_r^{(0)} = \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \Gamma(s) R^{-s} \left\{ \sin[s\pi/2 + \gamma] - \frac{B_k(s) \sin \pi s}{\sqrt{(2\pi)}} \right\} \mathrm{d}s.$$

In this form the integral converges absolutely if $c = \frac{1}{2}$ but in order to separate into two integrals it is necessary to write in the form

$$\phi_r^{(0)} = \int\!\!\!\!\!\!\rightarrow_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \Gamma(s) R^{-s} \left\{ \sin[s\pi/2 + \gamma] - \frac{B_k(s) \sin \pi s}{\sqrt{(2\pi)}} \right\} \mathrm{d}s$$

and then show that

$$I_s = \int\!\!\!\!\!\!\rightarrow_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \Gamma(s) R^{-s} \sin(s\pi/2)\, \mathrm{d}s, \quad I_c = \int\!\!\!\!\!\!\rightarrow_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \Gamma(s) R^{-s} \cos(s\pi/2)\, \mathrm{d}s$$

both exist. Here we consider I_s only, as the argument would be identical for I_c.

Consider therefore $I_s = \lim_{\varepsilon \to 0}(I_1 + I_2)$, where

$$I_1(\varepsilon) = \int_{c}^{c+\mathrm{i}\infty} \mathrm{e}^{\mathrm{i}\varepsilon s} \Gamma(s) R^{-s} \sin(s\pi/2)\, \mathrm{d}s, \quad I_2(\varepsilon) = \int_{c-\mathrm{i}\infty}^{c} \mathrm{e}^{-\mathrm{i}\varepsilon s} \Gamma(s) R^{-s} \sin(s\pi/2)\, \mathrm{d}s$$

and define

$$I_1^*(\varepsilon) = \int_{c}^{c+\mathrm{i}\infty} \Gamma(s) R^{-s} \sin s(\pi/2 - \varepsilon)\, \mathrm{d}s, \quad I_2^*(\varepsilon) = \int_{c-\mathrm{i}\infty}^{c} \Gamma(s) R^{-s} \sin s(\pi/2 - \varepsilon)\, \mathrm{d}s.$$

Then

$$I_1(\varepsilon) - I_1^*(\varepsilon) = -\int_{c}^{c+\mathrm{i}\infty} \Gamma(s) R^{-s} \mathrm{e}^{\mathrm{i}\pi s/2} \sin \varepsilon s\, \mathrm{d}s,$$
$$I_2(\varepsilon) - I_2^*(\varepsilon) = \int_{c-\mathrm{i}\infty}^{c} \Gamma(s) R^{-s} \mathrm{e}^{-\mathrm{i}\pi s/2} \sin \varepsilon s\, \mathrm{d}s.$$

Because $\int_c^{c+\mathrm{i}\infty} \Gamma(s) R^{-s} \mathrm{e}^{\mathrm{i}\pi s/2}\, \mathrm{d}s$ and $\int_{c-\mathrm{i}\infty}^{c} \Gamma(s) R^{-s} \mathrm{e}^{-\mathrm{i}\pi s/2}\, \mathrm{d}s$ both converge uniformly (with exponential decay $-\pi|s|$), it can be readily shown that

$$\lim_{\varepsilon \to 0}[I_1(\varepsilon) - I_1^*(\varepsilon)] = \lim_{\varepsilon \to 0}[I_2(\varepsilon) - I_2^*(\varepsilon)] = 0.$$

From a table of transforms (e.g. [1, Art. 10.29]), we have

$$I_1^* + I_2^* = 2\pi \mathrm{i} \mathrm{e}^{-R \sin \varepsilon} \sin(R \cos \varepsilon).$$

Hence I_s exists as an Abel summable integral and, after similar consideration to I_c it follows, on letting $\varepsilon \to 0$, that

$$\phi_r^{(0)} = \sin(R + \gamma) - \int\!\!\!\!\!\!\rightarrow_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \Gamma(s) R^{-s} \frac{B_k(s) \sin \pi s}{\sqrt{(2\pi)}}\, \mathrm{d}s$$

as anticipated. Using the definition of ϕ_r in Eq. (6), the $\sin(R + \gamma)$ terms cancel and we are left with the required representation of ϕ_r as the Abel summable divergent integral in Eq. (10).

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